Time Dependent Diffusion Coefficient of Zooplankton in an Interactive Environment

Hasika Wickrama Senevirathne and Chandani Dissanayake
Sri Lanka Technological Campus,
SLT Satellite Station, Padukka, Sri Lanka. e-mail:hasikak@sltc.edu.lk

***Abstract*—Various population dynamic models are being developed to describe the dynamics of a simple predator-prey system. The lowest level of food chain in a water habitat is the producer (Algae) and grazer *(Daphnia)* interaction. The behaviour of planktons in an interactive environment can be described by reaction-diffusion dynamic model. The diffusion part of the model governs the mass transport of species by diffusion. In many population dynamics models, the diffusion coeffcient of planktons are considered as a constant. Our work shows that the diffusion coefficient of zooplankton *(Daphnia)* depends on time. It behaves linearly with time. Further, this study is extended to investigate the behavior of diffusion coefficient under spatially heterogeneous environment using the FokkerPlanck equation with probability density function of behavior of zooplankton.**

***Index Terms*—Diffusion coefficient, dynamic models, Fokker-Planck equation, linear relationship, planktons**

I. Introduction

Planktons are diverse collection of microscopic organisms in large bodies of water and unable to swim against a current. These organisms are phytoplankton, zooplankton, bacterioplankton and mycoplankton. This scheme divides the plankton community into broad producer, consumer and recycled groups. Hence, they play a vital role in food chains as producer-grazer interaction. The dynamics of producergrazer interaction with algae (phytoplankton) as the producer and *Daphnia* (zooplankton) as the grazer in an interactive environment are considered in this work.

There are many population dynamic models of two interacting species being developed based on various aspects. They describe the behavior of population densities of two species in a food chain. These models are important to understand the changes of growth of species. Reaction- diffusion model is one of the dynamic models used in applications. The study of [1] has described population dynamics of predator (grass) - prey (zebra) interaction using reaction-diffusion system. The diffusion coefficient can be defined as the quantity of a material that in diffusing from one area to another passes through each unit of cross section per unit of time when the volume-concentration gradient is unity [2]. Using Fick's first law, flux is proportional to the gradient concentration of the material. In our study, J can be thought of as the number of animals.

$$J \propto -\frac{\partial u}{\partial x} \Rightarrow J = -D\frac{\partial u}{\partial x}, \qquad (1)$$

where D is the diffusion coefficient and dimesnsion can be identified as area per time. Most of the applications have considered the diffusion coefficient as constant for various purposes. However, diffusion coefficient for dynamic models in an interactive environment is not constant according to the literature review. Recent study [3] has explained the pattern formation in reactions-diffusion models of two chemical reactions with spatially inhomogeneous diffusion coefficients. The model has been analyzed in the case of a step function in space with a single point of discontinuity diffusion coefficient. The study [4] has described the estimates of vertical eddy diffusivity in the ocean interior. Eddy diffusivity can be modeled as a decreasing function of buoyancy frequency.

Furthermore, the horizontal advection is a process of observed distributions of planktons acting to mix traces with longer reaction times R_t. The diffusivity coefficient for a reactive tracer is a function of reaction time [5].The study [6] has described the analytical solutions for the two and three dimensional advection-diffusion equation. They have considered the transport of pollutants in a large variety of environments.The solution of this system depends on the spatial variable velocity and diffusion coefficient. Here, it has been assumed that the velocity component is proportional to the distance and the diffusion coefficient is proportional to the square of the corresponding velocity component.Three dimensional diffusion coefficient has been modeled using space variables in the directions of x, y, and z as

$$\begin{aligned} D_x &= D_0 u_0 x_2 \\ D_y &= D_0 v_0 y_2 \\ D_z &= D_0 w_0 x^2, \end{aligned} \qquad (2)$$

here D_0 is constant diffusion coefficient and u_0, v_0, w_0 are the constant velocities along x, y and z directions, respectively. The study [7] has explained the movements of *Daphnia* with a turning angle using experimental data. It has been neglected any interaction with neighbors or external fields to identify the *Daphnia* motion. The model for diffusion coefficient of single *Daphnia* with real time and space [7]

$$4D = \frac{1+\gamma}{1-\gamma}\frac{\lambda^2}{\tau}, \qquad (3)$$

where γ is the angular correlation with a Gaussian distribution having 20^0 mean and 36^0 standard deviation, τ is the time interval, and λ is the step length (10/3 mm). Hence, the diffusion coefficient of *Daphnia* in single movement is based on space and time variables. Avgar *et al.* [8] has been explained that the environment of resources and consumer interaction, consumer diffusion rates have been extremely sensitive to movement directionality.Furthermore, if resources travel exactly the same constant speed as the consumer, the diffusion coefficient (D) was expressed in [8] as

$$D = \frac{\pi \nu}{16 \rho r \left(1 + \frac{\pi}{8} \rho r \nu h\right)}, \qquad (4)$$

where ν,ρ and r are the speed of the consumer, resource density and the product of the effective radius, respectively. The paper [9] has described the method to model search time as a function of movement in a prey-predator system. First passage time is the time required for a random variable, such as an animals location in space, to go from a given starting point to a pre-defined endpoint. It has used red-fox to develop the model and this model has been valid for any ecological system. Spatial variability in movement speed has been incorporated into the model allowing the diffusion coefficient to vary in space. It has been built an ordinary differential equation for mean first passage time with spatially dependent diffusion coefficient [9]

$$D(x) \frac{d^2 T}{dx^2} + 1 = 0, \qquad (5)$$

where, T is the first passage time and $D(x)$ is the spatially variant diffusion coefficient. In addition to that, Turchin [10] has used the forward Fokker-Planck equation related to animal movement to the probability of the animal occurring at a particular point in time and space.

With the motivation from the literature review, the objective of this study is to model the diffusion coefficient of zooplankton with time mathematically.

This paper is organized as follows. In Section II, model development of diffusion coefficient of zooplankton with time is discussed. Further, the discussion and conclusion of the model development are presented in Section III. Finally, future work to improve this model is discussed in Section IV.

II. Model Development

A reaction-diffusion model can be used to describe population dynamics of predator-prey systems. The model is a system of two coupled partial differential equations in the form of [11]

$$\frac{\partial u_i(\underline{x}, t)}{\partial t} = f(u_i) + \nabla . \left(D_i . \nabla u_i\right), \qquad (6)$$

where $u_i(\underline{x},t)$ is the population density of planktons, $f(u_i)$ is the reaction part of the species and D_i is the diffusion coefficient of the species. Here, i = 1,2 refer producer and grazer, respectively.

According to the previous studies, the diffusion coefficient of zooplankton depends on various parameters.

This section is dedicated to present time dependent diffusion coefficient of zooplankton in an interactive environment. We refer one dimensional form of (6) to describe population dynamics of zooplankton.

A. Model the diffusion coefficient of zooplankton (Daphnia) in time

The study [12] has presented two different movement mechanisms.They may be integrated into predictions of consumer population spread as resource and competitor densities vary in four cases.

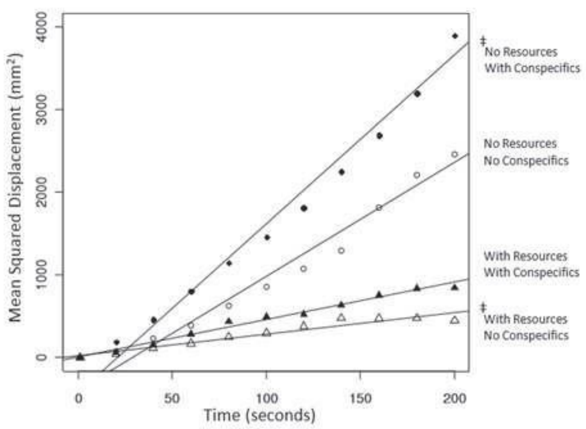

Fig. 1. Mean squared displacements of *Rotifers* at consecutive 20 second time intervals under four experimental treatments in 4 cases as with resource-with conspecifics, with resources - no conspecifics, no resources-with conspecifics, no resources- no conspecifics [11].

For our study, we consider the case with resources and no conspecifics in Fig. 1.

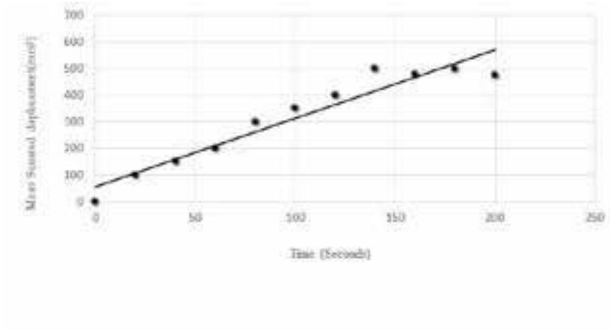

Fig. 2. Time versus mean squared displacement for the case with no resources and with conspecifics

A few of linear and nonlinear models are refitted to identify the dependency of diffusion coefficient of zoo-plankton on time. The best fitted model for data points in Fig. 1 is selected for further work.

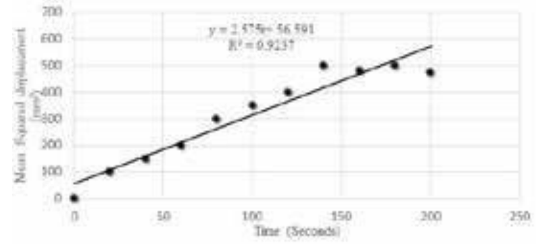

Fig. 3. The linear model for experimental data in Fig.1 gives relationship between time and mean squared displacement with the equation, $y = 2.57t + 56.591$

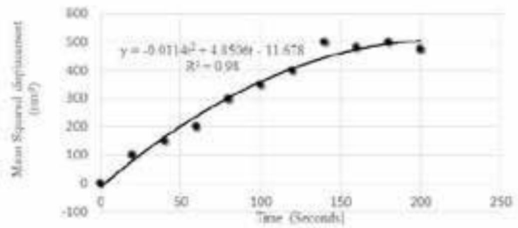

Fig. 4. The quadratic model for experimental data in Fig.1 gives relationship between time and mean squared displacement with the equation, $y = -0.0114t^2 + 4.8506t - 11.678$

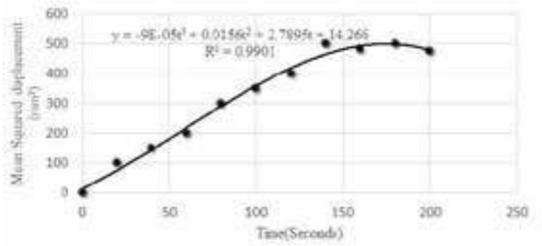

Fig. 5. The cubic model for experimental data in Fig.1 gives relationship between time and mean squared displacement with the equation, $y = (-9E-$
$0.5)t^3 + 0.0156t^2 + 2.7895t + 14.266$

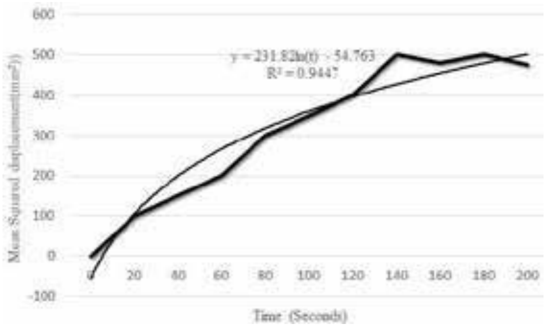

Fig. 6. The logarithm model for experimental data in Fig.1 gives relationship between time and mean squared displacement with the equation, $231.82\ln(t) - 54.763$

It has been considered the R-squared value and, parsimonious model for the above models. R-squared is measured how much of the variability can be explained by the model. Parsimonious model can be defined as a model that accomplishes a desired level of explanation or prediction with as few predictor variables as possible [13] [14].

Observing and comparing above figures according to the R-squared values and the parsimonious models, Fig. 4 shows the best fitted model. This gives the quadratic equation (7) as follows.

$y = -0.0114t^2 + 4.8506t - 11.678$, (7) where y - mean squared displacement (mm^2), t - time (s). In the study [12], it is mentioned that the rate of spread of a population, diffusion coefficient, is proportionate to the rate of change of mean squared displacement over time. Therefore, slopes of tangent lines of this best fitted model gives the diffusion coefficient. Using Equation (8), a model for diffusion coefficient (D) can be derives as

$$D = -0.0228t + 4.8506 . \qquad (8)$$

III. Discussion and Conclusion

In this study, our objective is to model the diffusion coefficient for zooplankton *(Daphnia)*. We accomplished this through two main steps. First, we identified space and time dependency diffusion coefficient by studying previous studies. Second, we built a model for diffusion coefficient of zooplankton which depends on the time variable using the experimental results of [12].

Planktons are generally microscopic organisms. Phytoplankton and zooplankton can be considered as the primary level of a food chain in the water system. In this study, it was considered reaction-diffusion model to describe population dynamics of Algae (phytoplankton) and *Daphnia* (zooplankton).The diffusion coefficient is the magnitude of the flux through a surface per unit concentration gradient out of plane. In our case flux is the number of animals. The rate of spread of populations of organisms is proportionate to the rate of change of the mean squared displacement over time [12]. The zooplankton used in [12] is *Rotifer*. Assuming similar behavior for *Daphnia*, we got the result. After analyzing experimental results of [12], we brought the relationship between time and diffusion coefficient. We were able to show that diffusion coefficient of *Daphnia* behaves linearly with time (explicitly).

The diffusion coefficient of zooplankton may depend on space. Previous studies mainly have mentioned that FokkerPlanck equation can be used to model the diffusion coefficient of zooplankton based on space [12] [15] . Using Taylor series method and graphical interpretation, [9] has built a model for first time passage time for red-fox. In their work, spatially dependent diffusion coefficient has been used. However, it can be used the backward Fokker-Planck equation to derive the first passage time directly [16]. Kuefler *et al.* [17] has used graphical interpretation of modeling the diffusion coefficient using the maximum likelihood approach. Each independent trail generated a time series of population proportions at different distances from initial point of origin and then it has been fitted a one-dimensional Fokker-Planck diffusion model. Fokker- Planck equation is a powerful tool to build the models statistically with effects of fluctuations [18].It has used Fokker-Planck equation to model a lot of applications such as sub dynamics of financial data [18], distribution of sunspot number [19] and the discrete kinetic models [20]. They have used Fokker-Planck equation to model the distributions of dynamic models because Fokker-Planck-equation is a vital role to formulate a generic choice of driver function for dynamic models. The next step of our study is to model the spatially variant diffusion coefficient for resource-grazer dynamic model. Hence, Fokker-Planck equation would be useful to derive the diffusion coefficient in spatially heterogeneous environment.

Logical linkage between functional response models is used in [8] to derive the diffusion coefficient of a consumer as functions of resource abundance, speed of consumer and the spatial aggregation. The functions of diffusion coefficients in [8] are the functions of resource density, consumer speed and effective radius. Therefore, the equations in [8] can be used to derive spatially varies diffusion coefficient.

As the conclusion of this work, the diffusion coefficient of zooplankton *(Daphnia)* behaves linearly with time. Further, the spatially varies diffusion coefficient can be modelled using Fokker-Planck equation and the mean squared displacement of consumers explicitly.

IV. Future Work

This work can be extended to various directions. The next step would be modelling the diffusion coefficient of zooplankton in space explicitly. It can be proposed that the diffusion coefficient of zooplankton for spatially non-homogeneous environment can be derived using Fokker-Plank equation. Hence, it has to be identified the probability density function varies with space and time of zooplankton in an interactive environment. Furthermore, the diffusion coefficients of planktons can be modelled with other factors such as environmental conditions (climatic and weather changes).In this work we considered one dimensional reaction -diffusion model. Another extension of this work would be modeling the diffusion coefficient considering higher dimensions. Moreover, numerical simulations can be done for reaction-diffusion system with the time dependent diffusion coefficient.

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